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STRATEGIC INTERACTIONS IN EDUCATION: THRESHOLD-TRIGGERED COMPENSATION OF COMPLEMENTARY INPUTS

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Strategic Interactions in Education: Threshold-triggered compensation of complementary inputs

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Abstract

Interactions between teachers and students depend on the compensation scheme for their complementary contributions to production of education. Should inputs be rewarded by constant unit prices, strategic complementarity would induce students to study more when better paid teachers work harder. In reality students and teachers are rewarded discretely when their joint output exceeds the threshold for passing exams and completing degrees. In a simple certainty setting where some agents precommit their input, and in plausible configurations of a more realistic setting where simultaneous input choices continuously determine the probability that joint output exceeds a threshold, inputs are strategic substitutes: students work less if teachers work harder, and *vice versa*. This insight makes it possible to characterize theoretically efficient threshold-triggered compensation schemes and helps interpret the often inefficient outcomes observed not only in education, but also when coauthors of academic papers or athletes in team sports are discretely rewarded for their joint publications or victories.

[+] Università di Torino, CEPR, CESifo, [▼] Università del Piemonte Orientale, LABORatorio R.Revelli. We gratefully acknowledge helpful comments received from participants at seminars in Parma and Torino and at the 64th annual conference of the Italian Economic Society. All errors remain our own. Movies about moviemaking are often awarded Oscars by the Academy of movie-makers, so we hope economists who teach for a living might be intrigued by our work and offer additional very welcome comments.

1 Introduction

Schools supply a service that needs much more client cooperation than a haircut or a theater performance, because students are an input as well as an output container in the production of education. Learning requires work that, like teaching and any other job, is paid. Education conveys private benefits that, in expected discounted terms, equal or exceed the labor income paid by the job students could now and later perform in the labor market.[1] Production of education also employs the work of teachers, whose effort and commitment are nowadays increasingly spurred by families eager to protect youngsters from failure and discomfort ("snowplow" or "bulldozer" parenting) as well as by employers who compete with each other to attract students.[2] Even though high dropout rates may result from efficient updating by students who revise expectations of the benefits and costs of education and of their own ability (Bertola, 2024; Collins and Lundstedt, 2024), it is often viewed as an indication of low teaching quality. Hence, education providers frown on teachers with high failing rates, and sometimes punish them harshly.[3] When teachers who stick to their standards may be blamed for their students' poor performance, there are obvious incentives to inflate grades and pass low-effort students who view themselves as customers who expect a degree in exchange for their fees (Molesworth et al, 2009).[4]

Pressure from parents and employers certainly heightens teacher burnout and job-related anxiety. Whether it actually improves learning is not as clear. Because production of education uses complementary teacher and student inputs, the marginal productivity of those inputs is

[1] Each year of schooling on average increases subsequent labor income by about 9% in the Psacharopoulos and Patrinos (2018) review of 1120 estimates across 139 countries and 65 years. Educational output also conveys benefits for society as a whole: one additional year of education reduces the probability of being convicted for property or violent crime by about 12% in the US (Lochner and Moretti, 2004), and increases the average GDP growth rate by about 0.37 percentage points in a panel of 50 countries over 40 years (Hanushek et al. 2008).

[2] In the UK for instance, the Office for Students increasingly relies on the teaching quality evaluation metrics based on students' responses to the National Student Survey (NSS) to shame degree programs with high dropout rates and punish the teachers involved.

[3] Professor Maitland Jones Jr., a contract teacher with decades of previous experience, was fired by New York University in the spring of 2022 after 82 of his 350 students filed a petition complaining about the low and frequently failing grades of his organic chemistry course (New York Times, October 3 2022; Princeton Alumni Weekly, December 2022).

[4] In a survey of university staff run in the UK, "46% of interviewed academics reported being pressured to mark students' work generously", "37% did not believe teaching was valued by their institution, 43% said they did not feel supported in their job, and 47% said they did not get enough support at work "(The Guardian, May 18 2015)

higher when teachers and students work harder. Hence, stronger work incentives for teachers should elicit more work from students as well as from teachers, and increase educational output. In reality, however, employers rely mostly on the binary signal sent by prospective workers' degrees and qualifications, and pay little if any attention to transcripts and other continuous indicators of educational achievements (Costrell, 1994, and its references), and teachers are also rewarded discretely when many of their students pass exams. Discrete rewards make teacher-student interactions more conflictual than cooperative, and their actions tend to be strategic substitutes: because rational students' work incentives are weaker when teachers help them more, and vice-versa, stronger teacher incentives reduce the students' input to an extent that may actually decrease the amount of human capital produced. Similar mechanisms operate in many other team-production situations with discrete rewards and complementary inputs: strategic substitutability is natural among coauthors of research papers they would like to be accepted for publication, siblings who cooperate in taking care of elderly parents, or prosecutors seeking to indict and convict a suspect.

We characterize these strategic interactions first in a baseline model where students (or teachers) commit their input with certainty and then, adding team production to Bertola's (2024) model of performance in random pass-or-fail assessments, in a more realistic setting where random production allows inputs to be chosen simultaneously. We assume throughout that the educational output is objectively (if imprecisely) assessed by a third party, leaving to further research the possibility that teacher might rationally be lenient to avoid being punished for their students' performance.

The remainder of the paper is organized as follows. Section 2 presents the paper's contributions against the background of the relevant literature. Section 3 sets up a simple Cobb-Douglas educational technology and, focusing on a single teacher-student pair, models the behavior of teachers and young adult students who rationally supply their complementary inputs to joint education production under different compensation schemes.[5] With continuous rewards, inputs are strategic substitutes. If discrete rewards are instead triggered by a pass threshold, inputs are strategic substitutes. Section 4 adapts Bertola's (2024) characterization of individual incentives in random pass-or-fail assessments to let exam performance depend jointly and imprecisely on the teacher's and student's inputs. As in Kono and Yagi (2008) and Bertola (2024), the equilibrium strategies depend non-monotonically on the amount of noise in the system. When

[5] A pedagogical approach where empathy, reciprocity, and motivation shape behavior and outcomes may also be relevant, especially for primary education. However, even children need effort to learn, react to incentives, and can behave in surprisingly manipulative strategic ways.

exams are sufficiently precise, agents "fear failing" and supply inputs above the level that would be optimal under certainty. In this configuration, choices are strategic substitutes. When the amount of noise is sufficiently large to imply failure rates above 50%, agents "hope to pass" and supply inputs below the certainty level. Here, choices are strategic complements. Section 5 shows that the outcome of these interactions can be the solution of a principal-agent problem involving school principals who manage teachers and students as production agents. Section 6 concludes summarizing the model's implications and outlining the limitations and possible realistic extensions of the modelling framework.

2 Related literature and contribution

The present paper contributes to different streams of economic research. On the one hand, it is substantively related to work that studies how student/teacher interactions and the design of teacher incentives affect educational outcomes. On the other hand, it is methodologically close to the contributions to personnel economics that analyze optimal incentives in teams. The remainder of this section reviews the two literatures and outlines how our modelling framework sheds new light on strategic behavior and incentive schemes in education and other team-production settings where agents working together individually receive a discrete prize if their joint output exceeds a threshold, including preparation of research grant applications, paper submissions, and team sports.

2.1 Economics of education

The literature on the paper's topic includes contributions to the economics of education that model interactions between teachers and schools on the one hand, and students and their families on the other. Several papers point out that teachers' and students' inputs may be complements or substitutes. Correa and Gruver (1987) show that students respond differently to an increase in teachers' input (and harder grading) depending on the elasticity of substitution between educational success and non-academic activities in their utility functions and between their inputs and their teachers' in the education production function. De Fraja and Landeras (2006) and De Fraja, Oliveira, and Zanchi (2010) allow the reaction functions of schools and families to be upward or downward sloping at the Nash equilibrium input choices, without modeling explicitly why, and show by comparative statics methods that increasing teachers' incentives improve education production only when inputs are complements, consistently with the results presented in what follows. While these papers do not model how exams assess educational achievements, Bertola (2021) and its references study how endogenous student effort may be elicited by fear of failing without any role for teachers.

Other contributions model and study empirically the role of teachers' incentives. Holmstrom and Milgrom (1991) caution against the perverse effects of merit pay focused on only some indicators of multi-taking teachers' educational performance, but do not model students as rational agents. Barlevy and Neal (2012) show that such negative effects may not arise under more sophisticated incentive schemes that link teachers' compensation to the rank of their students within comparison sets. The scant evidence on the effect of teacher incentives is unsurprisingly inconclusive for pay-for-performance (Contreras and Rau, 2012) and for evaluation systems (Bleiberg et al., 2021).

The present paper's model contributes new theoretical insights adopting linear utilities and an educational technology with complementary inputs and unitary elasticity of substitution, and shows that discrete rewards triggered by pass-or-fail exams allow strategic substitutability to arise from pre-committed inputs when performance evaluation is noiseless, or from sufficiently precise exams when randomness coordinates inputs to levels exceeding those that would be chosen under certainty. In "fear-failing" equilibria, both teachers and students err on the side of caution and provide inputs above the level that would ensure passing if the other agent's and the exam outcome were known with certainty. In this configuration, inputs are strategic substitutes: since agents fear that negative estimation errors may trigger a fail, more input by the teacher (or an easier exam) decreases the probability of failing at each student's input level, making it optimal for the latter to reduce her contribution and vice-versa. In another configuration both the teacher and the student "hope-to-pass" and provide inputs below the level that would suffice to pass under certainty. Because they hope that positive estimation errors may trigger a pass despite low input, more input from one agent (or an easier test) increases the marginal effect of one's contribution on the pass probability and elicits more input from the other: the equilibrium features strategic complementarity.

2.2 Personnel economics and tournament theory

The proposed model's team-production structure is similar to that assumed in personnel economics papers where team members may be paid shares of observable output as well as threshold-triggered prizes. Holmström (1982) shows how a principal can elicit efficient, unobservable, imperfectly substitutable inputs from team members by rewarding them by shares of the output and imposing penalties or paying rewards when their output falls below or exceeds a threshold;

Kono and Yagi (2008) model within-team tournament incentives with normal noise when team members compete for a discrete prize that is awarded only to the best contributor; hybrid models, such as that of Gershkov, Li, and Schweinzer (2009), let team members receive a share of the output as well as discrete prize based on a noisy measure of their contribution's ranking.

Should it be possible to pay teachers and students shares of output, and not just discrete prizes, they would care about how much they produce as well as about whether the threshold is passed, and the model would be a log-linear version of Holmström (1982). But discrete rewards are realistic not only in higher education, as students mostly aim at completing degrees and teachers are to some extent rewarded if they do, but also in other settings where passing an evaluation triggers rewards for workers. Such labor contract tournaments are useful because it is administratively easier to compensate agents by prizes rather than in proportion to measured output (Lazear and Rosen, 1981). Moreover, contracts that motivate agents who work together in settings where both individual and team output are observable must reconcile a trade-off between team and individual incentives as well as relative and absolute performance (Irlenbusch and Ruchala, 2008; Danilov, Harbring, and Irlebusch, 2019). These models often assume additively separable production functions, but also acknowledge that the assumption is not neutral: Gershkov, Li, and Schweinzer (2009) for instance show that acceptable sharing rules are always efficient in equilibrium when output is a linear function of inputs, not when inputs are imperfectly substitutable.

This paper's model lets joint output be the only observable measure of performance, so different prizes cannot be awarded to different agents conditional on their relative outputs, and realistic input complementarity plays a crucial role in the characterization of equilibrium efficiency. The extreme assumption that agents only care about whether output exceeds the pass threshold, not about its level, has an equally extreme and intriguing implication from the point of view of a principal who would like inputs and output to maximize the surplus generated by a team of agents who supply complementary inputs to joint production: because these agents either fail or succeed together, their contingent rewards should be of equal size. In education, this is more a theoretical curiosity than a usable prescription, as the assumptions of this very stylized model and the organization of education are both distant from a reality where efficiency is infrequent.

3 Education technology and compensation systems

Theoretical results applicable to the other complementary-input production processes mentioned above and to production teams with more than two members are most simply derived by focusing on one teacher-student pair engaged in production of education.

Denote with y the economic value of education, a function of the student's input x_S and of the teacher's input x_T. It is realistic and will be substantively important in what follows to suppose that function to be supermodular, so that the marginal productivity of each input is increasing in the level of the other. A decreasing-returns Cobb-Douglas production function

$$y = (x_S)^{\alpha_S} (x_T)^{\alpha_T}, \; \alpha_S + \alpha_T < 1 \tag{1}$$

has that property, and a simple form that makes it possible to obtain closed-form solutions. All results should be qualitatively valid for other twice continuously differentiable and strictly concave supermodular production functions.

Both the student and the teacher bear fixed costs, $\bar{\mu}_S$ and $\bar{\mu}_T$, and variable costs that it will be convenient to assume are linear: each unit of x_S costs γ_S, and each unit of x_T costs γ_T. Because $y > \gamma_S x_S + \gamma_T x_T$, the value of education exceeds the cost of variable inputs. The education technology is not viable if this surplus falls short of the $\bar{\mu}_S$ and $\bar{\mu}_T$ intercepts of the cost functions and any other production costs. Otherwise, the technology can be activated, and the Appendix shows that the surplus is maximized when

Remark 1 *The input ratio and output level are efficient at*

$$\frac{x_T^*}{x_S^*} = \frac{\alpha_T}{\alpha_S} \frac{\gamma_S}{\gamma_S} \tag{2}$$

$$y^* = \left(\left(\frac{\alpha_S}{\gamma_S} \right)^{\alpha_S} \left(\frac{\alpha_T}{\gamma_T} \right)^{\alpha_T} \right)^{\frac{1}{1-(\alpha_S+\alpha_T)}}. \tag{3}$$

In the models that follow market and contractual relationships may but need not support this efficient outcome.

3.1 Linear compensation

The input choices that produce education could in principle be decentralized by prices in a market where the student and the teacher both take as given the payment of each unit of their input as well as the other's input choice. Market-like linear payments per unit of input could be either implemented directly by exchanges between the student and the teacher (as in the case of private tutoring paid by the hour by a student who values each unit of learning), or by an institution that offers such contracts to students and teachers. In this perfectly competitive equilibrium, total payments to variable inputs are $\omega_S y$ for the student and $\omega_T y$ for the teacher if ω_S and ω_T denote their respective shares of output. Hence, as the Appendix shows,

Remark 2 *If the student and teacher are rewarded continuously with linear shares of their joint output, their equilibrium inputs are*

$$\hat{x}_S = \left(\left(\frac{\omega_S}{\gamma_S} \right)^{1-\alpha_T} \left(\frac{\omega_T}{\gamma_T} \right)^{\alpha_T} \right)^{\frac{1}{1-(\alpha_S+\alpha_T)}}, \ \hat{x}_T = \left(\left(\frac{\omega_S}{\gamma_S} \right)^{\alpha_S} \left(\frac{\omega_T}{\gamma_T} \right)^{1-\alpha_S} \right)^{\frac{1}{1-(\alpha_S+\alpha_T)}} \tag{4}$$

which are at the efficient level if $\omega_S = \alpha_S$ and $\omega_T = \alpha_T$.

With linear costs, variable inputs earn no inframarginal surplus. This simplifies computation and interpretation of the results. Derivations would be considerably more complicated, but qualitatively similar, if upward sloping supply functions generate inframarginal surplus for the variable factors. The competitively supplied inputs in (4) matter only for efficiency, not for distribution, and determine the residual surplus

$$y - y\omega_S - y\omega_T = (1 - \omega_S - \omega_T) \left(\left(\frac{\omega_S}{\gamma_S} \right)^{\alpha_S} \left(\frac{\omega_T}{\gamma_T} \right)^{\alpha_T} \right)^{\frac{1}{1-(\alpha_S+\alpha_T)}}$$

available to cover fixed costs and to reward any residual-claimant third party. Contractual arrangements could in principle implement this linear compensation scheme, and would allow the education technology (3) to operate efficiently if they equate the student's and teacher's share of output to their partial output elasticities.

3.2 Passing a threshold...

In real-life classrooms there is no market that prices inputs by the unit, and education itself is not valued as a continuous variable: students may pass or fail examinations that grant a standard qualification that is valued by the labor market and may be needed for a job. As in families or firms, however, rational economic behavior is very relevant even though choices are not rewarded by linear prices. Still focusing on one exam and a teacher-student pair, suppose the student passes and obtains λ_S if the educational output's value y exceeds an exogenous threshold

ζ. And suppose the teacher is also rewarded discretely if the student passes (or, equivalently, punished if the student fails). Denoting μ_T the fixed component of the teacher's pay, the teacher earns $\mu_T + \lambda_T$ if $y \geq \zeta$, just μ_T otherwise.

Exceeding the pass threshold is a waste of input, so each of the players supply the minimum input that ensures passing. When each input is just enough to obtain a pass given the other, then the Appendix shows that

Remark 3 *If the student and teacher are rewarded discretely when their joint output exceeds a threshold ζ, their reaction functions*

$$\bar{x}_S \equiv (x_T)^{-\frac{\alpha_T}{\alpha_S}} \zeta^{\frac{1}{\alpha_S}}, \ \bar{x}_T \equiv (x_S)^{-\frac{\alpha_S}{\alpha_T}} \zeta^{\frac{1}{\alpha_T}} \tag{5}$$

coincide along a continuum of possible Nash equilibria such that $\zeta = (x_S)^{\alpha_S} (x_T)^{\alpha_T}$.

In a threshold triggered compensation scheme where the student and teacher choose their input simultaneously, the pass threshold determines the amount of education produced in each of the many possible equilibria of this chicken game in continuous strategies. In the absence of contractual obligations, precommitment or credible signaling may select one of these.

3.3 ...when one input is precommitted

Consider the case where the student can credibly commit the input (we briefly discuss below the symmetric implications of assuming that the teacher can commit instead). Taking as given the student's committed input, the teacher supplies the Nash equilibrium input from (5) if the contingent payment from bringing the student to passing competence exceeds the total input cost required to do so. Knowing this, the student optimally chooses the minimum level of input that elicits the teacher's participation, squeezing the teacher's surplus to zero. The Appendix shows that

Remark 4 *If the student and teacher are rewarded discretely when their joint output exceeds a threshold ζ and the student input can be credibly committed, the inputs' equilibrium levels are*

$$\tilde{x}_S = (\gamma_T/\lambda_T)^{\frac{\alpha_T}{\alpha_S}} \zeta^{\frac{1}{\alpha_S}}, \ \tilde{x}_T = \lambda_T/\gamma_T \tag{6}$$

which are at the efficient level if $\zeta = y^*$ and $\lambda_T = \alpha_T \zeta = \alpha_T y^*$.

In equilibrium the exam is passed and the educational output's value is exactly ζ. Out of equilibrium, the exam is not passed and $y = 0$, because zero variable inputs are the rational choice conditional on not passing. The teacher's input is the premium for bringing the student to passing competence divided by the teacher's unit input cost. Since the student supplies just enough input to make that outcome preferable to zero input for the teacher, the teacher participates if the base pay μ_T (that would be paid when failing, and providing no input) is larger than the outside option $\bar{\mu}_T$.

Given the teacher's participation, the student's input choice depends only on the threshold and the parameters that determine the teacher's choice. The student participates if the pass reward λ_S exceeds the variable cost

$$\gamma_S \tilde{x}_S = \gamma_S \left(\gamma_T / \lambda_T\right)^{\alpha_T / \alpha_S} \zeta^{1/\alpha_S} \tag{7}$$

by at least the amount of tuition plus the fixed cost $\bar{\mu}_S$. In the efficient equilibrium, the student's variable cost (7) is $\alpha_S y^*$, the efficient share of total output that implies efficiency in the linear pricing scheme described in Remark 2: when $\zeta = y^*$ and $\lambda_T = \alpha_T y^*$ the student participates if the contingent reward λ_S is at least as large as the payment the student receives in the efficient equilibrium with continuous rewards.

Given participation, setting the threshold ζ at the efficient output y^* in (3) and the teacher's incentive pay at $\lambda_T = \alpha_T \zeta = \alpha_T y^*$ elicits the efficient inputs from both the teacher and the student and maximizes the total surplus. Should the teacher move first, symmetric reasoning indicates that efficiency requires $\lambda_S = \alpha_S y^*$: only a portion of the human capital's value should accrue to the student upon passing.

3.4 Complementarity and substitutability of educational inputs

Any $\{x_S, x_T\}$ pair that produces $y = (x_S)^{\alpha_S} (x_T)^{\alpha_T}$ can be a decentralized equilibrium of either a compensation scheme where the teacher and student are paid linear shares of their joint output or of a game with committed student (or teacher) input where fixed and variable payments can be set to elicit participation and possibly approach efficiency. These different compensation schemes determine how the precommitted student adjusts supply to a change in the teacher's payment (and vice-versa should the teacher input be precommitted). The Appendix shows that

Remark 5 *In the linear compensation scheme, an increase in teacher's payment increases the student's input,* $d\hat{x}_S/d\omega_T > 0$. *In the threshold-triggered reward scheme with precommitted student input, an increase in teacher's payment decreases the student's input,* $d\tilde{x}_S/d\lambda_T < 0$.

In the linear compensation system with precommitted student input, when ω_T becomes

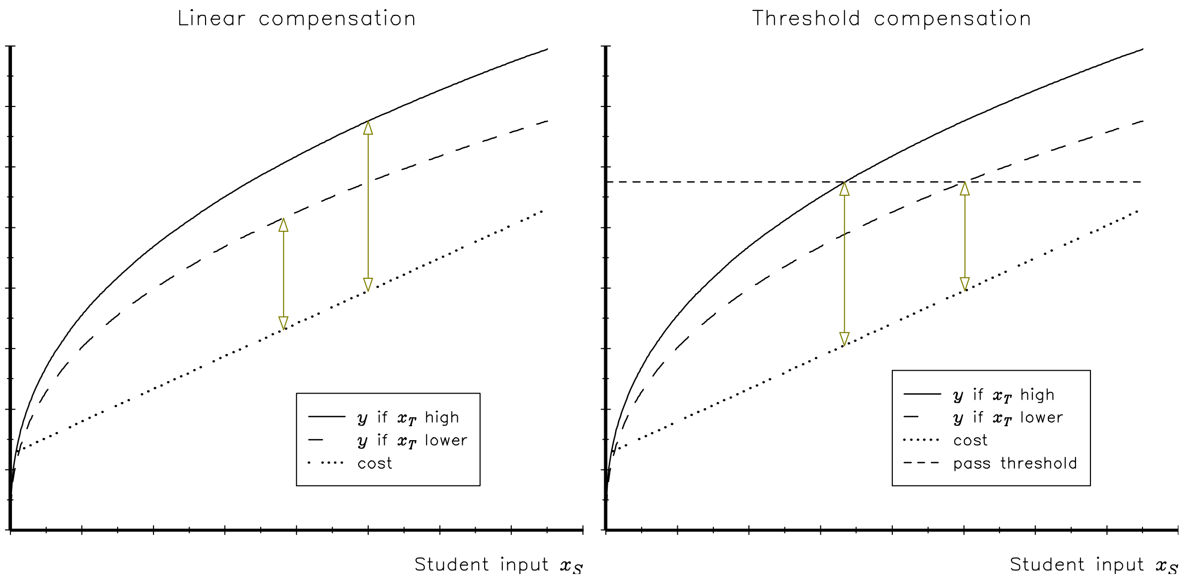

Figure 1: The concave lines plot joint output at given teacher input as a function of student input. The student's optimal variable input maximizes the excess of the benefit over cost, indicated by the arrows.

larger, average productivity y/x_T must be lower to obtain $\gamma_T = \omega_T y/x_T$. Under decreasing returns to scale this is brought about by increasing x_T: a teacher who receives a receives a larger share of output works harder. As shown in the left-hand panel of Figure 1, the higher x_T implied by a larger ω_T symmetrically increases the marginal productivity of the complementary input the student chooses so as to equate its marginal contribution to joint production to the slope of the linear cost function, which has a positive intercept representing the opportunity and monetary cost of participation $\bar{\mu}_S$. If the unit payment of either variable inputs is larger, both inputs increase: they are strategically as well as technologically complementary.

In the threshold-triggered compensation system, a higher teacher's reward λ_T instead reduces the student's input (6), as in the right-hand panel of Figure 1. When the teacher is rewarded more strongly for a pass outcome, and in equilibrium works harder, then the student needs to works less to ensure passing. Inputs are complementary in technology, but strategic substitutes.

3.5 Credible precommitment in real-life classrooms

The chicken-game in continuous strategies defined by the reaction functions (5) has a well-defined Nash equilibrium if either of the two agents can credibly precommit their input. In real-life classrooms, students may do so (and force teachers to do the necessary work) by ostentatiously not taking notes, not reading the textbook, not doing homework assignments. In some private schools students expect to do little work and get a lot of help, which they do obtain because teachers are monitored and rewarded for pass outcomes, hence prepare detailed slides and deliver clear lectures. In other settings, teachers may precommit their input by coming in late and mumbling, and generate output with minimal teaching and a lot of student effort that, in this certainty equilibrium, does ensure that the exam is passed. Senior scholars and prosecutors may also minimize their own precommitted input and extract from less experienced and reputable colleagues the work needed to publish papers and win trials. Similarly, siblings who establish a reputation of unreliability can force their brothers and sisters to take care of elderly parents.

These equilibria illustrate an interesting insight, but their reliance on commitment strategies limits their realism and applicability. Because the reaction functions (5) are discontinuous, a pure-strategy simultaneous-moves equilibrium does not exist under certainty. The problem can in reality to some extent be solved by rules that coordinate inputs. Teachers cannot skip classes or tell jokes instead of covering the syllabus without facing some punishment, and students can be required to hand in some solutions to problem sets and to attend lectures and be silent, even though they cannot be forced to do their homework and pay attention.

More importantly, the compensation systems modeled above rely on excessively exact information. Not only are exams imprecise, but it is difficult for students to assess the expected future value of additional individual labor income, and teachers are compensated in complicated and delayed ways for their input's contribution to the human capital of the students who pass their exams. The next Section shows that when these sources of uncertainty are described by continuously distributed random variables then input choices are continuous functions of each other, and support well-defined simultaneous-move equilibria.

4 Random exam results

In reality teachers and students need not know exactly how their inputs determine their joint output, and are aware that exams estimate it imprecisely. What follows modifies the model to let inputs bring output above the threshold and trigger a reward not with certainty, but only with some probability. In the constant-elasticity production framework introduced above it is convenient to let uncertainty influence outcomes multiplicatively. Suppose $y = (x_S)^{\alpha_S} (x_T)^{\alpha_T} e^{\varepsilon}$, where ε is a random variable that summarizes all relevant sources of uncertainty.[6] We rule out

[6]As in Bertola (2024) if each input's productivity (and possibly the passing threshold) are random, and so is the exam's outcome at given output, only the randomness implied by the total of all zero-mean errors around rational expectations influences the choices of interacting agents who are aware of their

precommittment, which would have the same implications as under certainty, and suppose that the teacher and the student choose their input simultaneously.

The student passes when y exceeds a threshold ζ. The probability of that event depends on the student's input x_S according to

$$\begin{aligned}\text{Prob}\left(\alpha_S \ln x_S + \alpha_T \ln x_T + \varepsilon > \ln \zeta\right) &= 1 - \text{Prob}\left(\varepsilon < \ln \zeta - \alpha_S \ln x_S - \alpha_T \ln x_T\right) \\ &= 1 - F\left(\left(\ln \bar{x}_S - \ln x_S\right) \alpha_S / \sigma\right),\end{aligned}$$

where σ is the standard deviation of ε and $F(x) \equiv \text{Prob}(\varepsilon/\sigma \leq x)$ its zero-mean standardized distribution function, which we will assume Gaussian in what follows, and

$$\ln \bar{x}_S \equiv \frac{\ln \zeta - \alpha_T \ln x_T}{\alpha_S} \tag{8}$$

is the student log input that under certainty would just suffice, as in (5), to pass at given teacher's input.

4.1 Input choices

The input chosen by a student at given $\ln \bar{x}_S$, which by (8) depends on the teacher's input x_T, maximizes the probability of obtaining the reward λ_S net of the input cost, solving

$$\max_{x_S} \left(1 - F((\ln \bar{x}_S - \ln x_S)\, \alpha_S/\sigma)\right) \lambda_S - \gamma_S x_s.$$

For a differentiable distribution function the first-order condition

$$F'((\ln \bar{x}_S - \ln x_S)\, \alpha_S/\sigma) \frac{\alpha_S \lambda_S}{\sigma x_S} - \gamma_S = 0 \tag{9}$$

identifies a local maximum when the left-hand side's derivative with respect to x_S is negative. There can be more than one local maximum when an increasing marginal cost schedule crosses the density of additive errors more than twice (Bertola, 2024). The functional forms we assume rule out that possibility and the resulting complications, which would not qualitatively change the insights we derive in what follows. When $F'(\cdot)$ is the zero-mean Gaussian density, the Appendix shows that condition (9) has either no or two real positive solutions. There is no solution if the marginal product of x_S in terms of expected pass reward is always below its marginal cost, hence the optimal choice is at the zero-input corner solution. When there are two solutions, the larger one satisfies the second order condition and has a neat explicit form, which

common imperfect information.

in what follows eases derivation and illustration of its interesting properties. The Appendix shows that

Remark 6 *If the student and teacher are rewarded discretely when their ramdomly perturbed joint output exceeds a threshold* ζ*, the student's optimal choice given the teacher's can be positive or zero. When it is positive, the parameters may or may not imply that it exceeds the certainty Nash level,* $\ln x_S - \ln \bar{x}_S > 0$*.*

Adopting the same terminology as in Bertola (2024), where a student or other assessee chooses how to perform a test without cooperating with a teacher or other agents, we shall say that when $\ln x_S - \ln \bar{x}_S > 0$ the student "fears failing"because of negative estimation errors. When $\ln x_S - \ln \bar{x}_S < 0$ instead, the student "hopes to pass", and will indeed pass only as a result of random errors. As we shall see, the former behavior is optimal when the exam is not very random, and leads rational agents to leave a margin of safety against negative estimation errors that may trigger a failure. When uncertainty looms large, conversely, it can be rational for rational agents to save input costs "try their luck" in the hope that positive errors will let them pass.

The parameter sets that make "fear-failing" or "hope-to-pass" behavior optimal also determine how the student responds to a change of the pass threshold and, in our setting, of the teacher's input taken as given by the student. The Appendix shows that

Remark 7 *When* $\ln x_S - \ln \bar{x}_S > 0$*, more input from the teacher (or an easier test) decreases the student's input,* $d \ln x_S / d \ln \bar{x}_S > 0$*. Conversely, when* $\ln x_S - \ln \bar{x}_S < 0$*, more input from the teacher (or an easier test) increases the student's input,* $d \ln x_S / d \ln \bar{x}_S < 0$*.*

When the student and teacher both err on the side of caution and supply inputs above the certainty level, choices are strategic substitutes. Conversely, when they try their luck and supply less input than under certainty would just suffice to pass, their choices are strategic complements. To see more intuitively why fear-failing choices imply strategic substitutability, rearrange (9) to read

$$F'((\ln \bar{x}_S - \ln x_S)\,\alpha_S/\sigma) = \frac{\gamma_S x_S}{\alpha_S \lambda_S} : \tag{10}$$

because the first-order condition equates the density of the log noise ε to an increasing function of x_S, a higher pass threshold shifts that density horizontally as in Figure 2, where it is easy to see that whether the student's choice of x_S increases or decreases in $\bar{x}_S$ depends on whether that density's slope is negative or positive. As in Kuhen and Landeras (2014) and Bertola (2024),

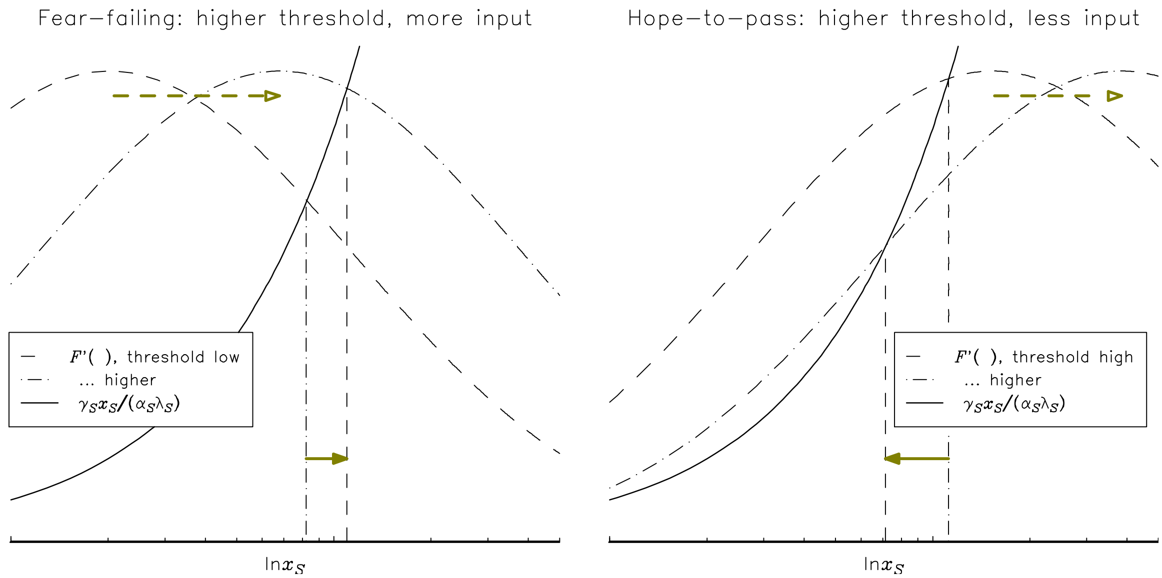

Figure 2: Plots on a logarithmic horizontal axis of the expression on the right-hand side of the rearranged first-order condition (10) and of possible normal densities on its left-hand side. Different pass thresholds, at the mode of the densities, shift the crossing points marked by vertical lines where (10) is satisfied.

this depends in turn on whether the student's optimal input is above or below the mode of the measurement error density, at the input (8) that under certainty would just suffice to ensure passing. When the student fears that negative errors may imply a fail, a higher $\bar{x}_S$ increases the probability of failing at each x_S, and the optimal x_S increases to offset some of this effect. Symmetrically, when $\ln x_S - \ln \bar{x}_S < 0$ and the student hopes that positive estimation errors will trigger a pass despite low input, a higher bar decreases the input's marginal effect on that probability and, as in the right-hand panel of Figure 2, makes it optimal to decrease x_S.

4.2 Equilibrium

A Nash equilibrium with positive inputs may exist at the intersection between the student's and teachers' reaction functions. The resulting system of nonlinear equations determines the equilibrium input levels as cumbersome solutions of quadratic equations. Deriving them is not more informative then interpreting plausible numerical solutions in light of the reaction functions' properties as we do in what follows.

In the two diagrams of Figure 3 the student's and teacher's reaction functions, plotted as dashed lines, cross when inputs are either both above or both below the solid line that plots the continuum of Nash equilibria that under certainty satisfy $\zeta = (x_S)^{\alpha_S} (x_T)^{\alpha_T}$ and where the

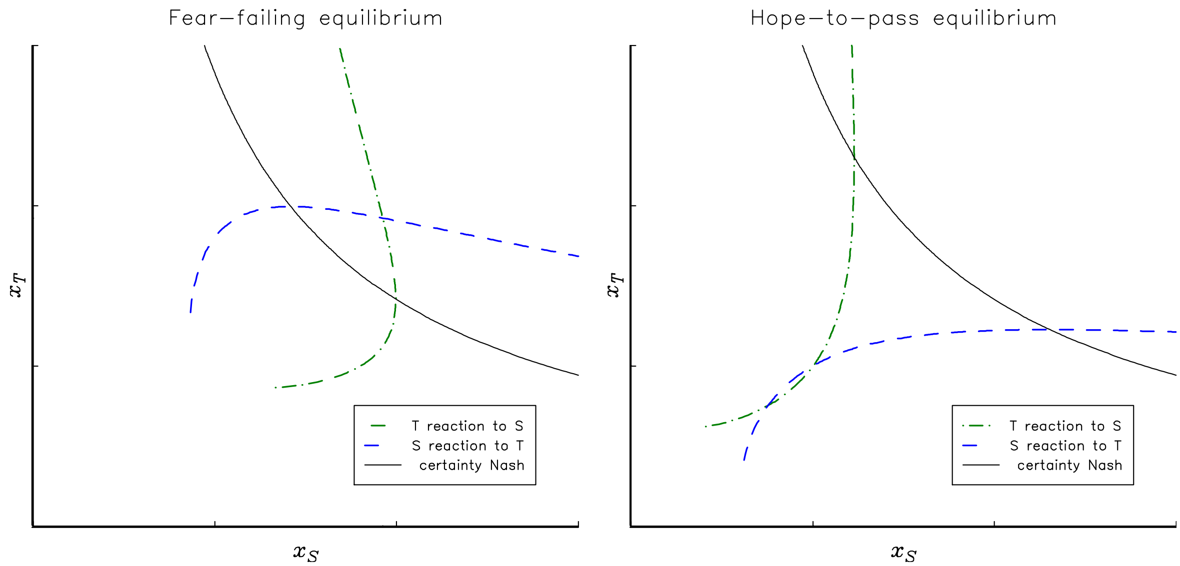

Figure 3: Strategic substitutability or complementarity of student and teacher input choices in equilibria where the pass probability is above 50% (left diagram) or below 50% (right diagram).

pass probability is 50%. Because the input chosen by fear-failing student increases when the passing threshold is more demanding, it declines when the teacher provides more input and the equilibrium is above the certainty Nash line, as in the left-hand diagram. Conversely, the passing bar is higher when the teacher supplies less input, and when it is so high as to bring the student's optimal input below the certainty Nash line, then the student's input decreases in the pass threshold, and increases in the teacher's input. Equilibria feature strategic substitutability when the teacher and the student fear failure, strategic complementarity when equilibrium inputs are chosen on the basis of hope-to-pass considerations.

The student's and teacher's reaction functions are respectively horizontal and vertical when they cross the certainty Nash line. There, a change in one of the agents' input leaves the other unchanged. Below it, lower inputs increase the effective pass threshold and have increasingly large negative implications for hope-to-pass choices: the induced negative variation in one of the agents' inputs tends to minus infinity as the other declines in the hope-to-pass region. Above the certainty Nash line the reaction functions have slopes of opposite signs and may cross at most once. Below the Nash line both reaction functions have positive slope, and there may be zero, one, or two positive-input hope-to-pass equilibria with failure rates over 50%.

When the optimal input of either agent ceases to be positive, the reaction functions slopes approach zero and infinity, and both inputs discretely jump to the zero-input corner solutions on the horizontal and vertical axis: then $\{x_S = 0, x_T = 0\}$, which is always a Nash equilibrium as there is no production when either input is zero. Conversely, as the Appendix shows,

Remark 8 *At any equilibrium with positive inputs, their ratio is*

$$\frac{x_T}{x_S} = \frac{\gamma_S}{\gamma_T}\frac{\alpha_T}{\alpha_S}\frac{\lambda_T}{\lambda_S} \tag{11}$$

which is at the efficient level (2) *if* $\lambda_S = \lambda_T$.

The inputs-ratio derived in (11) depends on relative rewards, costs, and output elasticities in sensible ways: the teacher's equilibrium input is relatively larger when it is less costly, or increases output more elastically, or is rewarded more generously. The ratio does not depend on σ, because the model's functional forms let variation of the pass criterion's randomness influence both inputs equally.

The student and teacher should for efficiency be rewarded with the same prize, regardless of technological asymmetries. This equal compensation result follows from the assumption that compensation is discrete and does not continuously depend on observed output. Because the student and the teacher randomly fail or succeed together, efficiency intuitively requires their stakes to be the same. We briefly discuss below reasons why this is difficult if at all possible to implement in education, but realistic in team sports and other formally similar situations.

4.3 Implications of randomness

The imprecision σ of the pass criterion plays an interesting role in determining whether a fear-failing input choice is optimal for the student (and, in equilibrium, for the teacher as well), in which case output is above the certainty Nash equilibria implied by the other parameters and there is strategic substitutability between the input choices.

Figure 4 plots a family of numerical solutions for a symmetric parameterization of the student's and teacher's problems and many values of σ. The diagram on the right in Figure 4 shows how fear-fail equilibria with strategic substitutability depend continuously on σ. For small σ the numerical solution is close to the center of the certainty Nash line, because the reaction functions are mirror images of each other and both tend to the certainty Nash as σ goes to zero.[7] While in the absence of commitment the equilibrium under certainty is indeterminate, even a very small σ coordinates inputs by making the payoffs a smooth function of strategic choices,

[7] The only nontrivial limit is $\lim_{\sigma\to 0}\left(\sigma^2 \ln(\sigma)\right) = \lim_{\sigma\to 0}\left(\frac{\ln(\sigma)}{\sigma^{-2}}\right)$, which by l'Hôpital rule equals $\lim_{\sigma\to 0}\left(\frac{\frac{1}{\sigma}}{-2\sigma^{-3}}\right) = \lim_{\sigma\to 0}\left(\frac{\sigma^2}{-2}\right) = 0$.

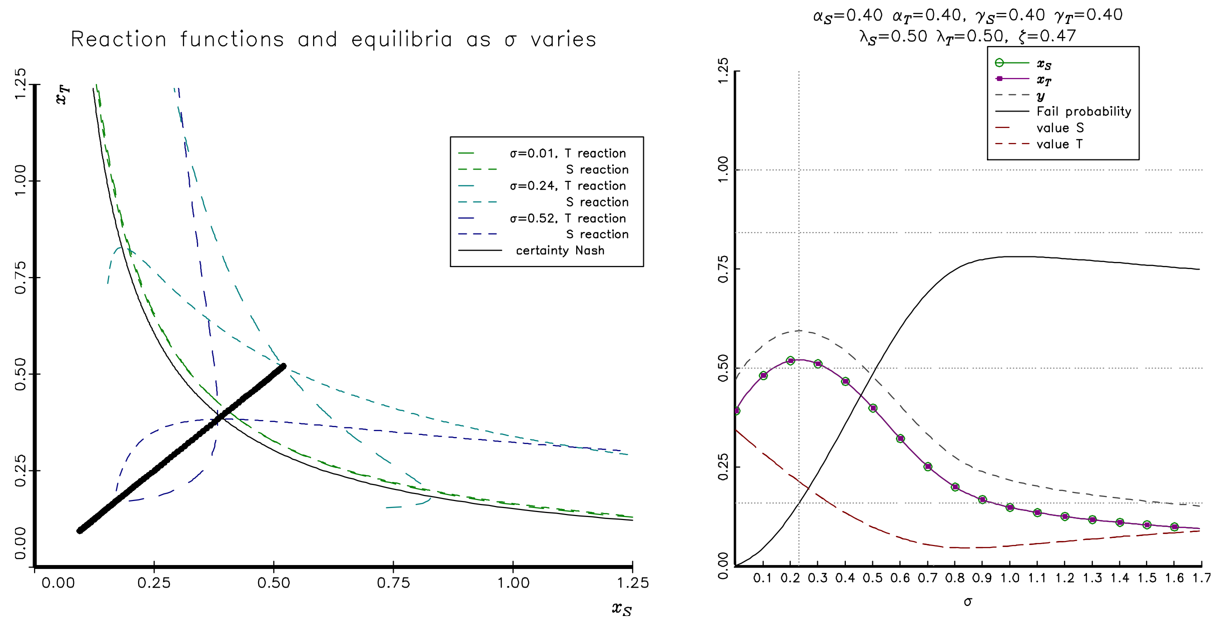

Figure 4: Equilibria as σ varies and reaction functions for σ values that have special implications, and $\sigma = 0.01$. On the right, plot of inputs, output, and fail probability as functions of σ.

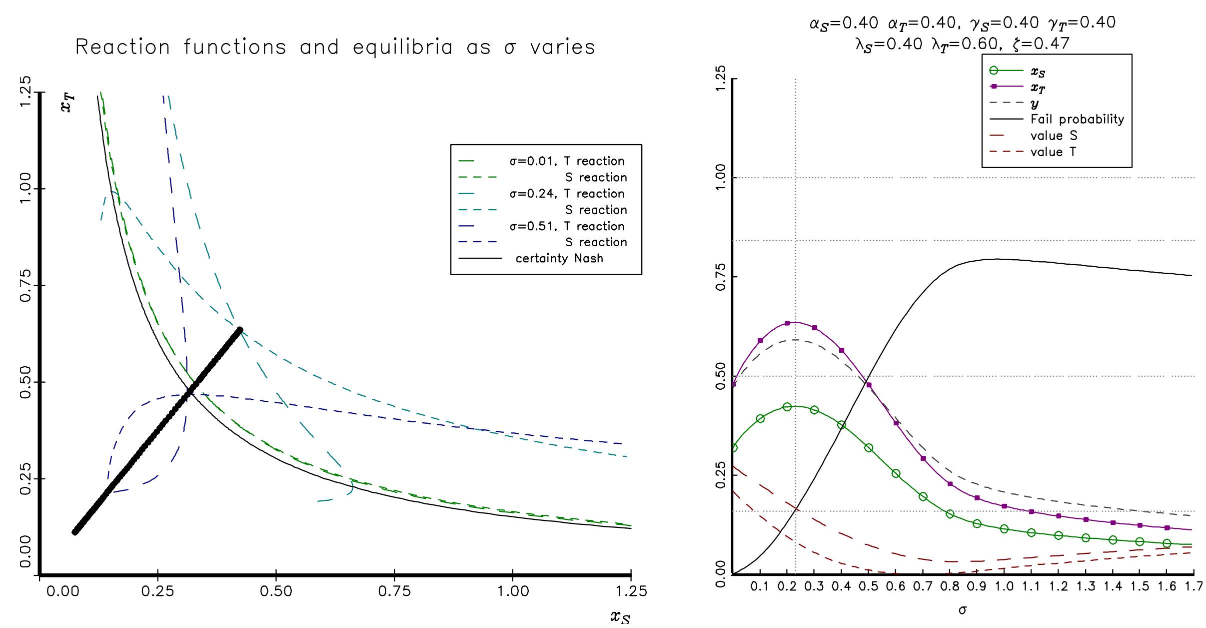

Figure 5: Asymmetric rewards, otherwise same as previous figure.

and supports a determinate equilibrium that can be arbitrarily close to a specific certainty Nash equilibria. In this continuous version of trembling-hand equilibrium refinement, as the noise becomes arbitrarily small (but positive) one of many equilibria is selected.

In Figure 4, as σ varies the equilibria all lie on a ray through the origin. Figure 5 shows a family of equilibria that are asymmetric because all technological parameters are the same for student and teacher inputs, but λ_S differs from λ_T. By (11), this and other asymmetries determine the slope of the linear locus along which σ moves the equilibria.

As σ increases from zero some terms of the reaction functions increase, some decrease, so the total effect is not monotonic. The right-hand diagram of Figures 4 and Figure 5 plot the student's and teacher's optimal inputs, their joint output, and the failure probability as functions of the evaluation's imprecision. As in Kono and Yagi (2008) and Bertola (2024), a more imprecise test initially heightens fear of failure, pushing both the teacher and the student to work harder despite the increased probability of failing at each chosen input level. They do so up to the point where their inputs reach a maximum and the failure probability is about 16%, which occurs when σ is slightly larger than 0.2 in both Figures. Then the reaction functions remain in the fear-failing region but the failure probability continues to increase as the inputs recede towards the Nash line. As σ increases and inputs decline, the failure probability approaches 50% and the equilibrium leaves the fear-failing region crossing the certainty Nash line, where the reaction curves are vertical and horizontal.[8]

A further increase of σ moves the equilibrium inside the hope-to-pass region, where input choices are strategic complements as long as their optimal value is positive. Larger values of σ decrease the equilibrium inputs, because strategic complementarity reinforces each agent's stronger incentives to save input costs and still hope to pass when the exam's outcome is more random: as the test outcome becomes increasingly less dependent on inputs, the failure probability declines towards 50% as σ approaches infinity. When σ is large enough to prevent existence of an interior solution for the input choices, both inputs drop to the corner solution at zero.[9] In the limit, the student and the teacher provide essentially no input but a pass outcome still occurs half the time.

[8]This happens at the value of σ identified by (A.7) in the Appendix, which is about 0.45 for the parameters used in plotting the figures.

[9]The critical value of σ solves (A.4) in the Appendix with equality. For some parameters, but not those used to plot the Figure, there are two positive solutions: then, equilibrium inputs are zero also over a range of σ values with a positive lower bound.

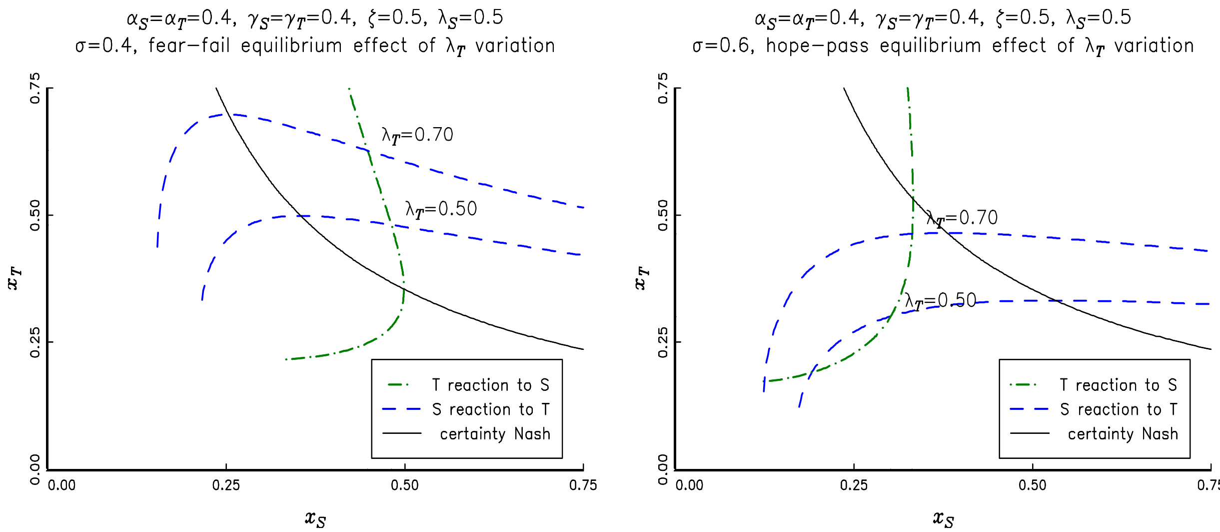

Figure 6: Equilibrium implications of teacher's reward variation.

4.4 Implications of compensation system parameters

At given σ, different values of λ_S and λ_T shift the student's and teacher's reaction functions, and change the slope of the ray along which equilibria are found in Figures 4 and 5. A larger teacher reward λ_T shifts the teacher's reaction functions upwards in Figure 6: a better-paid teacher works harder if pay depends on the student passing the exam, and supplies more input at given student input. How the student reacts depends on whether the equilibrium features strategic substitutability and fear-failing inputs, or complementarity and hope-to-pass inputs. In the left-hand diagram σ is small enough to imply a fear-failing equilibrium, so the student's input declines along an unchanged downward sloping reaction function. Should instead a large σ coordinate inputs at a low hope-to-pass level, then as in the right-hand diagram the student's input moves along a steep unchanged upward-sloping reaction function and increases, if only slightly, towards the certainty Nash equilibria locus. Because the student's reaction function is vertical at the certainty Nash line, when variation of the teacher's reward moves inputs across that line the student's input varies very little.

Reward changes can move the equilibrium below the Nash locus, and raise failure rates above 50% in hope-to-pass situations that are theoretically interesting, if not very realistic, and do illustrate the pitfalls of extreme imprecision: while small degrees of noise are desirable in that they may coordinate simultaneously provided inputs to levels that may be close to efficiency, as the next section briefly discusses, very inaccurate tests push teachers and students to reduce their inputs hoping that negative estimation errors will trigger a pass.

5 Implementation

We proceed to outline how principal-agent contractual arrangements may achieve efficiency in the equilibria characterized in the previous sections.

Consider a school that can examine the educational output; cannot monitor behavior but knows how it depends on the threshold ζ and teacher's incentive pay λ_T; must respect teacher and student participation constraints; and, like any private firm, maximizes profits and pays inputs only what is necessary to procure them. Unlike the goods and services produced and sold by other firms, the economic value of a school's educational output is by default embodied in students who choose how to supply their labor, and have no obligation to pay some of their resulting income to the school. The school can however appropriate some of this value by charging tuition to students, who are not only its customers but also the workers who supply some of the inputs needed to produce that output.

In general, the contingent reward λ_S is the portion of the educational output y that accrues to students who pass the exam and obtain higher labor income, more prestigious and less stressful work, and other private benefits. Under certainty, a precommitted student's surplus depends on λ_S,

$$\lambda_S - \gamma_S \tilde{x}_S = \lambda_S - \gamma_S \left(\gamma_T/\lambda_T\right)^{\frac{\alpha_T}{\alpha_S}} \lambda_S^{\frac{1}{\alpha_S}} \tag{12}$$

If the educational output's value is all embodied in the student then $\lambda_S = y$, and since the value of the education produced in these threshold-triggered equilibria under certainty is exactly $y = \zeta$, then $\lambda_S = \zeta$. The student participates if the resulting surplus, equation (A.9) in the Appendix, is no lower than the the outside option $\bar{\mu}_S$ plus the tuition charged by the school, which a profit-maximizing principal sets at the highest level compatible with enrolment and, for the school to be viable, should at least cover the teacher's variable input cost $\gamma_T \tilde{x}_T = \lambda_T$ and participation cost $\bar{\mu}_T$, as well as the cost of other production factors. If it exceeds those costs the school earns positive profits, which may be driven to zero by free entry in the school market.

What remains to be determined are the profit-maximizing choices of the teacher's contingent reward λ_T and of the exam's threshold ζ. The Appendix shows that

Remark 9 *The school profits are maximized when the teacher's incentive pay and pass threshold are at the efficient levels $\lambda_T = \alpha\zeta$ and $\zeta = y^*$.*

The λ_T and ζ parameters that the school chooses to maximize profits are the same that maximize the sum of the teacher's and student's surplus net of the teacher's incentive pay: intuitively, as residual claimant, the school that owns the education technology operates it efficiently, and charges a tuition that exhausts the excess of the value of education over the student's outside option.[10] Strategic substitutability implies that the student's surplus 12 is increasing in the teacher's reward: a precommitted student of a well-paid, hard-working teacher can appropriate the economic value of education supplying little input, and consequently, is willing to pay more for education. Knowing this, a profit-maximizing principal chooses the teacher's incentive pay as to maximize (12) net of $\bar{\mu}_T$. In this equilibrium any λ_T is in fact incentive-compatible from the teacher's viewpoint, whose variable surplus

$$\lambda_T - \gamma_T \tilde{x}_T = \lambda_T - \gamma_T (\lambda_T / \gamma_T)$$

is squeezed to zero by the student's ability to precommit. Note that if the owner of the education technology did not take as given how the market rewards the student but could choose $\lambda_S \neq \zeta$ the optimal choice would be $\lambda_S = \zeta$, which maximizes the surplus (12) and thus the tuition the student is willing to pay. For efficiency the agent who is able to precommit the input should appropriate the entire value of the human capital produced.

Should the teacher be able to precommit the input, the value of the student's equilibrium surplus would symmetrically be driven to zero, while the teacher's could be positive. To appropriate some of that value the principal should lease the education technology to the teacher, pay a scholarship $\bar{\mu}_S$ to ensure the student's participation, and could charge a rental fee equal to the excess of λ_T over the sum of the teacher's outside option $\bar{\mu}_T$ and variable input cost $\gamma_T (\gamma_S / \lambda_S)^{\alpha_S / \alpha_T} \lambda_T^{1/\alpha_T}$, and to maximize profits should maximize that surplus by setting $\zeta = y^*$ and $\lambda_S = \alpha_S \zeta$. The student's efficient graduation premium would fall short of the entire educational output, which should be contingently paid to the teacher and be equal to the human capital produced, $\lambda_T = y^*$: as before the precommitted agent should for efficiency appropriate the entire value of the educational output.

Under certainty, implementation by contracts is in theory simple and intuitive. Should the student pre-commit input, it would entail familiar tuition and wage payments. Should the teacher pre-commit instead, implementation would entail rather unusual arrangements whereby teaching jobs are auctioned off to individuals who do as little as possible. Under certainty, efficiency requires that the contingent reward paid upon passing to the agent who does not

[10] In the numerical example that generates the figures presented in the previous sections, $\alpha_S = \gamma_S = \alpha_T = \gamma_S = 0.40$ and $\bar{\mu}_S = \bar{\mu}_T = 0$ imply $y^* = 1.00$, and a profit-maximizing principal offers a contract $\{\mu_T,\ \lambda_T\} = \{0,\ 0.40\}$ to the teacher, sets the optimal threshold at $\zeta = y^* = 1.00$, and the optimal tuition at the 0.60 value of the student's surplus (12) net of the outside option $\bar{\mu}_S$, earning a positive profit of 0.20.

precommit be equal to the share of output that agent would receive in the efficient equilibrium with continuous rewards characterized in Remark 2.

Both situations are unrealistic but convey insights into how efficiency can be obtained in the more complicated case where randomness rather than precommitment drives the process of equilibrium selection. When the agents move simultaneously and pass outcomes are only randomly related to their performance, efficiency requires that each reward is equal to the average of the two contributions, and the equal compensation principle of Remark 8 follows: teacher contracts and student payments should aim at rewarding them equally for passing, which would ensure that the ratio of their inputs is at the efficient level. While under certainty the equilibrium output is pinned down exactly by the passing threshold ζ, which can readily be set at the efficient level, random pass outcomes can obtain a given production level through different combinations of the threshold, absolute rewards, and exam precision. In a fear-failing equilibrium where the teacher and the student are awarded some fixed amount if the realization of output is above the threshold, the equilibrium can intuitively produce an output level that is very close to the efficient y^* under certainty if σ is small.[11] These normative results might provide useful guidance to a school principal who designs the school staff's compensation scheme and the students' tuition and scholarships but, as we proceed to discuss in the next concluding Section, implementing equal compensation is much more difficult in education than in other team-production settings with complementary inputs and threshold-triggered rewards.

6 Summary and qualification of the results

Motivated by teacher performance incentives that focus on their students' exam results, the present paper has set up a simple team-production framework that allows for technological complementarity and restricts rewards to be discretely triggered by a pass threshold, and established that tightening work incentives for teachers would reduce the students' educational input should they be able to pre-commit their input, and would have symmetric implications should teachers' input be precommitted. The same strategic substitutability obtains when realistic randomness of exam results allows simultaneous input choices to reach a determinate equilibrium where, if exams are precise enough, students and teachers "fear-failing" and supply inputs above the level

[11] With the parameters used in producing the figures, if $\sigma = 0.01$ setting $\zeta = 0.980$ (2% percent below the efficient output $y^* = 1.00$) and $\lambda_S = \lambda_T = 0.490$ implies that the student's and teacher's inputs ($x_S = 1.005$ and $x_T = 1.005$) and expected output ($(x_S)^{\alpha_S} (x_T)^{\alpha_T} e^{\sigma^2/2} = 1.004$) are very close to their unitary efficient levels, and so is the resulting surplus.

that under certainty would just suffice to pass. Failure becomes less likely when teachers work harder, hence rational students work less when teachers are rewarded more strongly for pass-outcomes, and technologically complementary inputs are strategic substitutes. Larger amounts of noise increase the probability of failing at given input, up to a point where teachers and students find it rational to "try their luck" supplying inputs below the certainty level, hoping that positive estimation errors will trigger a pass. In such wasteful equilibria with very low pass rates case, inputs are strategic complements.

The modeling framework and results are applicable to other situations where agents supply complementary inputs and receive discrete prizes when their joint output exceeds a threshold. Readers may be particularly familiar with inefficient strategic interactions among coauthors who aim at publishing research papers as well as with those between teachers and students that motivate the paper. The model's assumptions and characterization offer useful insights, but the real world is as usual more complicated in a variety of respects. We proceed to discuss briefly the model's realism and possible extensions focusing on education, where efficiency would require all inputs to be rewarded equally and appropriately when their joint output discretely exceeds a threshold.

Implementing that efficient allocation would require school principals to have much more powerful instruments and more accurate information than what is available to them in a reality where the economic value of education is difficult to estimate, and most of it is embodied in the students' human capital. If the student's contingent reward is by default the private value of a pass outcome, for efficiency it would need to be shared between the student and with the institution that awards contingent compensation to the teacher. To some extent this may be implemented by taxing the labor income of a public university graduate, which reduces the student's pass reward to only a portion of its private value and allows the government to raise the revenue that rewards the teacher. Private institutions might somewhat less realistically rebate a portion of tuition if the exam is passed or, because alumni give gifts to private universities if they graduate but do not if they drop out, earn tax-like revenues similar to those of public schools. While such limited redistribution of contingent input rewards can hardly be fine-tuned to the extent required by efficient educational production, equal compensation is observed in other settings where rewards are triggered when joint output exceeds a threshold. In team sports salaries vary widely, but victory bonuses are typically the same for all team members, and coauthors are rewarded equally for joint publications in some research fields, including Economics.

When analyzing efficiency our partial-equilibrium model treats as choice variables rewards that in reality are determined by equilibrium feedbacks that the agents and principal need not internalize. The reward for passing exams and obtaining degrees is not determined by contractual arrangements, but by the information conveyed by exam results on students' productivity: should the structure of education production and exams be known, employers should pay no attention to passing or failing, as among identical students those who pass and those who fail really have provided the same input and obtained the same output. In reality pass or fail outcomes are obtained by heterogeneous students who expect (and possibly miss) different graduation premia, and obtain from exam results information about their own ability that can be more or less precise in different grading systems (Collins and Lundstedt, 2024). While the incentive effects of discrete rewards is always the same as characterized above, computing the relevant conditional expectations and assessing the efficiency of educational production would be exhausting even for economists who specify a model with known functional forms and parameters, and very difficult if at all possible also for market participants who have very limited information about the organization of education production and exams.

The education supported by discrete teacher compensation schemes can be too much, or too little, or too expensive also because of standard contractual and market inefficiencies (such as rent-seeking behavior, monopoly power, and externalities) that distort linear pricing as well as threshold-triggered compensation (Correa and Gruver, 1987; Gershkov, Li, and Schweinzer 2009). In education, the discrete-reward scheme's parameters need not be chosen by a residual claimant who aims at maximizing the social surplus. Students would like compensation parameters that let them perform almost none of the work needed to increase human capital to the level required by the exam, and if the parents or government agencies who pay cost of education yield to some such wishes that effortless education will expensively deviate from the efficient configuration. Education can be inefficiently expensive also when it gives too little incentive pay to the teachers, so that students get too little help and work too hard, as may be the case in public school systems where teachers have a powerful say on compensation schemes.

Several extensions of the model would not change the qualitative insights it provides, but deserve to be mentioned here and studied by further research. The model focuses on a single student-teacher pair and, for clarity and convenience, adopts a log-linear production function that yields closed-form solutions. In reality, as in Correa and Gruver (1987), elasticities of substitution are not unitary in education production by multiple students and multiple teachers. It is well understood that relative grading induce competition among students (Landeras, 2009), but a free-riding low-effort equilibrium is possible if the teacher is rewarded when students pass: should some students precommit low input and successfully obtain non-excludible help from the teacher and other students, incentives to work decline for all students.

Increasing the precision of exam results may be possible at a cost (Bertola, 2024, and its references). In this paper's setting, this can help avoid inefficient equilibria with hope-to-pass behavior, which however may be unavoidable if there is high exogenous uncertainty about the productivity of student and teacher inputs. It would also be interesting to relax the assumption that exams assess the educational output objectively, albeit imprecisely. Allowing teachers to play any part in determining the passing threshold that triggers their discrete rewards has obvious implications. As in macroeconomic policy models where central banks accommodate the inflation that price setters expect, inability to commit undermines credibility in educational settings where students know or learn that teachers aim to pass many students.[12] A teacher who decides whether the student passes and is punished for fail outcomes cannot credibly threaten to fail students who do not work, because it will be *ex post* optimal to pass them anyway. In the resulting time-consistent equilibrium students do not work, and the value of education must be very low.

[12]Teachers may be encouraged or even forced to indulge in grade inflation. Archaeology professor Paul Buckland lost his job when he realized that the marks he had given to his second-year students were systematically raised by his employer, the University of Bournemouth, and complained. In his words "degrees are rapidly becoming devalued. If nothing is required from the student in terms of effort or ability the "product"– the degree – is not worth anything "(The Independent, June 11 2012).

Appendix

Derivation of Remark 1

The technological total surplus

$$(x_S)^{\alpha_S}(x_T)^{\alpha_T} - \gamma_S x_S - \gamma_T x_T$$

is maximum when the marginal productivities $\alpha_S y/x_S$ and $\alpha_T y/x_T$ are equal to their technological unit costs γ_S and γ_T. Hence,

$$x_S^* = \alpha_S y^*/\gamma_S, \; x_T^* = \alpha_T y^*/\gamma_T$$

which implies the efficient ratio in (2). The efficient output solves

$$y^* = (\alpha_S y^*/\gamma_S)^{\alpha_S}(\alpha_T y^*/\gamma_T)^{\alpha_T},$$

which rearranges to (3). □

Derivation of Remark 2

In a perfectly competitive equilibrium the unit input prices $\omega_S y/x_S$ and $\omega_T y/x_T$ must coincide with the constant per-unit costs γ_S and γ_T that underlie their perfectly elastic supply functions, hence

$$x_S = \omega_S y/\gamma_S, \qquad x_T = \omega_T y/\gamma_T$$

which with (1) imply

$$x_S^{1-\alpha_S} = x_T^{\alpha_T}\omega_S/\gamma_S, \qquad x_T^{1-\alpha_T} = x_S^{\alpha_S}\omega_T/\gamma_T$$

Solving for x_S and x_T yields (4) in Remark 2. Their ratio

$$\frac{x_T}{x_S} = \frac{\omega_T}{\omega_S}\frac{\gamma_S}{\gamma_T}$$

is at the efficient level (2) if $\omega_S = \alpha_S$ and $\omega_T = \alpha_T$ as stated in the second part of Remark 2. □

Derivation of Remark 3

The teacher and student best choices are

$$\bar{x}_S = \arg\min x_S \text{ s.t. } (x_S)^{\alpha_S}(x_T)^{\alpha_T} \geq \zeta, \qquad \bar{x}_T = \arg\min x_T \text{ s.t. } (x_S)^{\alpha_S}(x_T)^{\alpha_T} \geq \zeta,$$

with solution (5) in Remark 3. □

Derivation of Remark 4

Solving $\min x_T$ s.t. $(\tilde{x}_S)^{\alpha_S}(x_T)^{\alpha_T} \geq \zeta$, the teacher satisfies the constraint with equality and supplies the Nash equilibrium input from (5) if $\lambda_T \geq \gamma_T \bar{x}_T$, zero otherwise. The student commits the minimum input that prevents zero input by the teacher by solving min x_S s.t., $\lambda_T \geq \gamma_T \bar{x}_T$, which, using (5), is

$$\lambda_T \geq \gamma_T \left(x_S\right)^{-\frac{\alpha_S}{\alpha_T}} \zeta^{\frac{1}{\alpha_T}}$$

Solving this for x_S yields the expression for $\tilde{x}_S$ in (6), while inserting this in (5) yields the expression for $\tilde{x}_T$ in (6). When $\zeta = y^*$ and $\lambda_T = \alpha_T y^*$ the teacher's input

$$\tilde{x}_T = \alpha_T y^* / \gamma_T \tag{A.1}$$

is at the efficient level and, because total output is y^*, so is the student's, at $\alpha_S y^*/\gamma_S$. In fact, inserting (A.1) in

$$\tilde{x}_S = \left(\tilde{x}_T\right)^{-\frac{\alpha_T}{\alpha_S}} \left(y^*\right)^{\frac{1}{\alpha_S}}$$

and equating the resulting expression to $\alpha_S y^*/\gamma_S$ yields

$$\left(\alpha_T y^*/\gamma_T\right)^{-\frac{\alpha_T}{\alpha_S}} \left(y^*\right)^{\frac{1}{\alpha_S}} = \alpha_S y^*/\gamma_S,$$

which can be rearranged to

$$\left(\alpha_T/\gamma_T\right)^{-\alpha_T} = \left(\alpha_S/\gamma_S\right)^{\alpha_S} \left(y^*\right)^{(\alpha_S+\alpha_T-1)}$$

to obtain (3). □

Derivation of Remark 5

Differentiating $\hat{x}_S$ in (4) w.r.t. ω_T yields

$$\frac{d\hat{x}_S}{d\omega_T} = \frac{\alpha_T}{(1-(\alpha_S+\alpha_T))\omega_T} \left(\left(\frac{\omega_S}{\gamma_S}\right)^{1-\alpha_T} \left(\frac{\omega_T}{\gamma_T}\right)^{\alpha_T}\right)^{\frac{1}{1-(\alpha_S+\alpha_T)}}$$

which is > 0. This proves the first statement in Remark 5. Differentiating $\tilde{x}_S$ in (6) w.r.t. λ_T yields

$$\frac{d\tilde{x}_S}{d\lambda_T} = -\frac{\alpha_T}{\alpha_S} \left(\frac{\gamma_T^{\alpha_T}\zeta}{\lambda_T^{\alpha_S+\alpha_T}}\right)^{\alpha_S}$$

which is < 0, as stated in Remark 5. □

Derivation of Remark 6

Inserting in (9) the Gaussian mean-zero standard density

$$F'\left(\left(\ln \bar{x}_S - \ln x_S\right)\alpha_S\right) = e^{-\frac{1}{2}\left(\frac{(\ln \bar{x}_S - \ln x_S)\alpha_S}{\sigma}\right)^2 - \ln\left(\sqrt{2\pi}\right)}$$

yields

$$e^{-\frac{1}{2}\left(\frac{(\ln \bar{x}_S - \ln x_S)\alpha_S}{\sigma}\right)^2 - \ln(x_S) - \ln\left(\sqrt{2\pi}\right)} \frac{\alpha_S \lambda_S}{\sigma} - \gamma_S = 0.$$

The derivative of the left-hand side with respect to $\ln(x_S)$,

$$\left(\frac{(\ln \bar{x}_S - \ln(x_S))\, \alpha_S}{\sigma} \frac{\alpha_S}{\sigma} - 1\right) e^{-\frac{1}{2}\left(\frac{(\ln \bar{x}_S - \ln(x_S))\alpha_S}{\sigma}\right)^2 - \ln(x_S) - \ln\left(\sqrt{2\pi}\right)},$$

is negative and the second-order condition is satisfied when

$$(\ln x_S - \ln \bar{x}_S) \frac{\alpha_S^2}{\sigma^2} > -1. \tag{A.2}$$

Rearranging and taking logs, (6) is a quadratic equation in $\ln x_S$:

$$-\left((\ln \bar{x}_S - \ln x_S)\, \alpha_S\right)^2 - 2\sigma^2 \ln x_S - 2\sigma^2 \ln\left(\frac{\sigma\gamma_S}{\alpha_S \lambda_S}\sqrt{2\pi}\right) = 0. \tag{A.3}$$

If the parameters and teacher's input are such that

$$\sigma^2 > 2 \ln\left(\frac{\sigma\sqrt{2\pi}\gamma_S}{\alpha_S\lambda_S}\right) \alpha_S^2 + 2\alpha_S^2 \ln \bar{x}_S > 0. \tag{A.4}$$

there are two real solutions,

$$\ln x_S = \ln \bar{x}_S \pm \frac{1}{\alpha_S^2}\left(\sqrt{\sigma^4 - 2\sigma^2 \ln\left(\frac{\sigma\sqrt{2\pi}\gamma_S}{\alpha_S\lambda_S}\right)\alpha_S^2 - 2\alpha_S^2\sigma^2 \ln \bar{x}_S} - \sigma^2\right) : \tag{A.5}$$

the smaller solution violates (A.2) and the larger solution identifies the logarithm of the student's optimal positive input. If the parameters do not satisfy (A.4) there is no real solution, and the zero input corner is optimal.

When (A.4) is satisfied, the optimal input is higher than the input the student would supply under certainty if $\ln x_S - \ln \bar{x}_S > 0$, i.e.

$$\sqrt{\sigma^4 - 2\sigma^2 \ln\left(\frac{\sigma\sqrt{2\pi}\gamma_S}{\alpha_S\lambda_S}\right)\alpha_S^2 - 2\left(\alpha_S\sigma\right)^2 \ln \bar{x}_S} > \sigma^2 \tag{A.6}$$

and lower otherwise. Squaring both sides of (A.6)

$$\sigma^4 - 2\sigma^2 \ln\left(\frac{\sigma\sqrt{2\pi}\gamma_S}{\alpha_S\lambda_S}\right)\alpha_S^2 - 2\left(\alpha_S\sigma\right)^2 \ln \bar{x}_S > \sigma^4$$

subtracting σ^4 from both and dividing the results by $2\alpha_S\sigma^2$ yields

$$-\ln\left(\frac{\sigma\sqrt{2\pi}\gamma_S}{\alpha_S\lambda_S}\right) - \ln \bar{x}_S > 0.$$

Using (8) for $\ln \bar{x}_S$, this condition is

$$\ln \sigma < -\frac{\ln \zeta - \alpha_T \ln x_T}{\alpha_S} - \ln\left(\frac{\sqrt{2\pi}\gamma_S}{\alpha_S\lambda_S}\right)$$

and rearranging yields

$$\sigma < \frac{\alpha_S \lambda_S}{\sqrt{2\pi}\gamma_S} \frac{(x_T)^{\alpha_T/\alpha_S}}{\zeta^{1/\alpha_S}} \tag{A.7}$$

a more explicit expression for the parameter sets that imply a "fear-failing" input choice higher than under certainty. □

Derivation of Remark 7

Differentiating (A.5) w.r.t. the effective bar threshold (8) yields

$$\frac{d \ln x_S}{d \ln \bar{x}_S} = \frac{\sqrt{\sigma^4 - 2\sigma^2 \ln\left(\frac{\sigma\sqrt{2\pi}\gamma_S}{\alpha_S\lambda_S}\right)\alpha_S^2 - 2\left(\alpha_S\sigma\right)^2 \ln \bar{x}_S} - \sigma^2}{\sqrt{\sigma^4 - 2\sigma^2 \ln\left(\frac{\sigma\sqrt{2\pi}\gamma_S}{\alpha_S\lambda_S}\right)\alpha_S^2 - 2\left(\alpha_S\sigma\right)^2 \ln \bar{x}_S}} \tag{A.8}$$

which is positive when (A.6) holds, and negative otherwise. This proves Remark 7. □

Derivation of Remark 8

At any equilibrium with positive inputs, $\ln x_S$ solves (A.3) with $\ln \bar{x}_S = (\ln \zeta - \alpha_T \ln x_T)/\alpha_S$,

$$-\left(\ln \zeta - \alpha_T \ln x_T - \alpha_S \ln x_S\right)^2 - 2\sigma^2 \ln x_S - 2\sigma^2 \left(\ln \sigma + \ln \sqrt{2\pi} + \ln\left(\frac{\gamma_S}{\alpha_S \lambda_S}\right)\right) = 0$$

and $\ln x_T$ solves a similar quadratic equation where the teacher's parameters γ_T,α_T,λ_T replace γ_S,α_S,λ_S,

$$-\left(\ln \zeta - \alpha_S \ln x_S - \alpha_T \ln x_T\right)^2 - 2\sigma^2 \ln x_T - 2\sigma^2 \left(\ln \sigma + \ln \sqrt{2\pi} + \ln\left(\frac{\gamma_T}{\alpha_T \lambda_T}\right)\right) = 0.$$

Subtracting one from the other and dividing by $2\sigma^2$ yields

$$\ln x_T - \ln x_S = \ln\left(\frac{\gamma_S}{\alpha_S\lambda_S}\right) - \ln\left(\frac{\gamma_T}{\alpha_T\lambda_T}\right)$$

which is the same as (11) in Remark 8. This proves the first part of Remark 8. Comparing (2) with (11) proves the second part of Remark 8. □

Derivation of Remark 9

The school's revenue is the maximum possible tuition, which is the student's surplus net of participation cost

$$\zeta - \gamma_S \left(\gamma_T/\lambda_T\right)^{\frac{\alpha_T}{\alpha_S}} (\zeta)^{\frac{1}{\alpha_S}} - \bar{\mu}_S. \tag{A.9}$$

The school costs in equilibrium include the teacher's base and incentive pay, $\bar{\mu}_T + \lambda_T$, and possibly other fixed costs. The derivatives of profit with respect to ζ and λ_T are

$$\frac{d}{d\zeta}\left(\zeta - \gamma_S \left(\gamma_T/\lambda_T\right)^{\frac{\alpha_T}{\alpha_S}} (\zeta)^{\frac{1}{\alpha_S}} - \bar{\mu}_S - (\bar{\mu}_T + \lambda_T)\right) = 1 - \zeta^{\frac{1}{\alpha_S}-1} \frac{\gamma_S}{\alpha_S}\left(\frac{\gamma_T}{\lambda_T}\right)^{\frac{\alpha_T}{\alpha_S}} \tag{A.10}$$

$$\frac{d}{d\lambda_T}\left(\zeta - \gamma_S \left(\gamma_T/\lambda_T\right)^{\frac{\alpha_T}{\alpha_S}} (\zeta)^{\frac{1}{\alpha_S}} - \bar{\mu}_S - (\bar{\mu}_T + \lambda_T)\right) = \frac{\alpha_T}{\alpha_S}\frac{\gamma_S}{\lambda_T}\left(\frac{\gamma_T}{\lambda_T}\right)^{\frac{\alpha_T}{\alpha_S}} \zeta^{\frac{1}{\alpha_S}} - 1 \tag{A.11}$$

Setting (A.11) to zero yields

$$\left(\frac{\gamma_T}{\lambda_T}\right)^{\frac{\alpha_T}{\alpha_S}} \zeta^{\frac{1}{\alpha_S}} = \left(\frac{\alpha_T}{\alpha_S}\frac{\gamma_S}{\lambda_T}\right)^{-1}. \tag{A.12}$$

Setting (A.10) to zero and inserting (A.12) yields

$$1 = \zeta^{-1}\frac{\gamma_S}{\alpha_S}\left(\frac{\alpha_T}{\alpha_S}\frac{\gamma_S}{\lambda_T}\right)^{-1}$$

which simplifies to

$$\lambda_T = \alpha_T \zeta.$$

Inserting this in (A.11) and rearranging yields

$$\zeta = \left(\left(\frac{\alpha_S}{\gamma_S}\right)^{\alpha_S}\left(\frac{\alpha_T}{\gamma_T}\right)^{\alpha_T}\right)^{\frac{1}{1-(\alpha_S+\alpha_T)}} = y^*.$$

which proves Remark 9. □

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